Absence of magnetohydrodynamic activity in the voltage-driven sheet pinch

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We have numerically studied the bifurcation properties of a sheet pinch with impenetrable stress-free boundaries. An incompressible, electrically conducting fluid with spatially and temporally uniform kinematic viscosity and magnetic diffusivity is confined between planes at $x_1=0$ and 1. Periodic boundary conditions are assumed in the x_2 and x_3 directions and the magnetofluid is driven by an electric field in the x_3 direction, prescribed on the boundary planes. There is a stationary basic state with the fluid at rest and a uniform current $\boldsymbol{J}=(0,0,J_3)$. Surprisingly, this basic state proves to be stable and apparently to be the only time-asymptotic state, no matter how strong the applied electric field and irrespective of the other control parameters of the system, namely, the magnetic Prandtl number, the spatial periods L_2 and L_3 in the x_2 and x_3 directions, and the mean values $\overline{B_2}$ and $\overline{B_3}$ of the magnetic-field components in these directions. [S1063-651X(96)10509-2]

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I. INTRODUCTION

One of the basic configurations in magnetohydrodynamics (MHD) is the pinch, namely, an electrically conducting fluid confined by the action of an electric current passing through it. Gradients of thermal pressure arising in the confinement region, notably a sheet or a cylinder or torus, are balanced by the Lorentz force. For instance, plasma confinement in toroidal devices for controlled thermonuclear fusion, such as the tokamak, is based on the principle of the pinch.

Static pinch configurations are subject to various instabilities, which have been studied extensively [1,2]. Of special interest here are the tearing modes, which belong to the class of the finite-resistivity instabilities [3,4]. By destroying magnetic surfaces, they can shorten the confinement time of fusion plasmas. Tearing modes are also thought to play a role for the explosive release of magnetic energy in space and astrophysical plasmas, e.g., substorms in the terrestrial magnetosphere and solar flares.

Pinch configurations may be maintained by external voltages. Alternatively, pinchlike *dynamic* structures may come about in a variety of circumstances, for instance, by mechanically forcing together two volumes of magnetofluid containing oppositely directed magnetic fields. This is the basic scenario for one of the two main directions of reconnection theory (where ''reconnection'' is used as a synonym for the fast conversion of magnetic energy into kinetic and thermal energies, in a process for which the the violation of the frozen-in-field condition of ideal MHD is essential) [5–9]. The other main direction has concentrated on the evolution of resistive instabilities. A review of work along the lines of both approaches may be found in the monograph of Biskamp [10].

In addition to analyzing the linear stability of specified equilibria, pinch configurations have been studied by numerically simulating the full nonlinear MHD equations. In general, the simulations were started from near-equilibrium states [11–14], but also relaxations from broadband-noise initial conditions to certain quasiequilibrium states were studied [15,16].

Besides linear stability analysis and numerical simulation, a useful tool for gaining insight into the global solution structure of a dynamical system is provided by bifurcation analysis. The main objective of a bifurcation analysis is the determination of all attractors, i.e., of the set of possible time-asymptotic states for a given set of external system parameters. It is then imperative that dissipative (Ohmic and viscous) losses are compensated for by some kind of permanent external forcing: otherwise the only time-asymptotic state is the trivial one with the fluid at rest and no magnetic field. In many numerical MHD simulations such an external forcing, which may be imposed in the form of an explicit external force or via appropriate boundary conditions, is absent, so that altogether relaxations towards the trivial state are studied.

Furthermore, in a bifurcation analysis the equilibrium states have to be really stationary. By contrast, it is common to apply linear stability analysis, in particular tearing mode analysis, to approximate equilibria, namely, to states in which the fluid is at rest but the magnetic field diffuses away.

In general, the set of the attractors and the changes of its composition and of the character of single attractors (the bifurcations) can, if at all, only be explored by numerical means. Under certain conditions, however, center manifold theory [17–19] can be used to obtain a low-dimensional system of amplitude equations, valid close to a bifurcation point and asymptotically in time. Grauer [20], Chen and Morrison [21], and Wessen [22] used center manifold reduction to study the time-asymptotic states of tearing mode evolution. Related preceding studies are due to Maschke and Saramito [23,24].

Most relevant for the present paper is recent work by Shan, Montgomery, and Chen [25–29], who studied numerically the bifurcation properties of an incompressible voltage-driven cylindrical pinch with circular cross section, periodic in the axial direction. For increasing an externally applied axial electric field, which can be prescribed on the boundary,

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transitions were observed from static equilibria to stationary states with flow, characterized by paired helical vortices and helical distortions of the electric current (which is axially directed in the quiescent state). If the driving electric field is raised further, the helical stationary states in turn lose stability and eventually turbulent states are observed. Such a behavior is found for spatially uniform [25] as well as nonuniform [26,27] electrical conductivity.

In the present paper we report on a bifurcation study of an incompressible sheet pinch with spatially uniform electrical conductivity, driven by an electric field prescribed on the boundary. Somewhat surprisingly, and in contrast to the behavior of the cylindrical pinch, a static basic state with uniform current density proves to remain stable and apparently to be the only time-asymptotic state, no matter how strong the driving electric field and irrespective of the values of other system parameters.

In Sec. II, after introducing the governing equations and their normalization, we explain system geometry, boundary conditions, and forcing by the external electric field. Then, in Sec. III we describe our numerical method and the calculations and present the result. Section IV, finally, contains a brief conclusion.

II. BASIC EQUATIONS, SYSTEM GEOMETRY, AND FORCING

We start from the equations for a nonrelativistic, incompressible, electrically conducting fluid with constant material properties (cf., e.g., [30]),

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right)=\rho\nu\nabla^2\boldsymbol{v}-\boldsymbol{\nabla}p+\frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}, \quad (1)$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=\eta\nabla^2\boldsymbol{B}+\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}), \quad (2)$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}=0, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0, \quad (3)$$

where $\boldsymbol{v}$ is the fluid velocity, $\boldsymbol{B}$ the magnetic induction, ρ the mass density, p the thermal pressure, ν the kinematic viscosity, μ_0 the magnetic permeability in a vacuum, and η the magnetic diffusivity [$\eta=(\mu_0\sigma)^{-1}$, σ denoting the electrical conductivity]. No externally applied force appears in Eq. (1). Transforming to nondimensional quantities according to

$$\boldsymbol{x}\rightarrow\boldsymbol{x}/L, \quad t\rightarrow t\Big/\frac{L^2}{\eta}, \quad \boldsymbol{v}\rightarrow\boldsymbol{v}\Big/\frac{\eta}{L},$$

$$p\rightarrow p\Big/\frac{\rho\eta^2}{L^2}, \quad \mathbf{B}\rightarrow\mathbf{B}\Big/\frac{\eta}{L}\sqrt{\mu_0\rho}, \quad (4)$$

Eqs. (1) and (2) become

$$\frac{\partial \boldsymbol{v}}{\partial t}=-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}+P_m\nabla^2\boldsymbol{v}-\boldsymbol{\nabla}p+(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}, \quad (5)$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla^2\boldsymbol{B}+\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}), \quad (6)$$

where P_m is the magnetic Prandtl number

$$P_m=\frac{\nu}{\eta}. \quad (7)$$

The transformations for the electric field $\boldsymbol{E}$ and the electric current density $\boldsymbol{J}$ ($=\boldsymbol{\nabla}\times\boldsymbol{B}/\mu_0$ in dimensional units), corresponding to the above normalizations, are

$$\boldsymbol{E}\rightarrow\boldsymbol{E}\Big/\frac{\eta^2}{L^2}\sqrt{\mu_0\rho}, \quad \boldsymbol{J}\rightarrow\boldsymbol{J}\Big/\frac{\eta}{L^2}\sqrt{\frac{\rho}{\mu_0}} \quad (8)$$

and the nondimensional Ohm law reads

$$\boldsymbol{J}=\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}. \quad (9)$$

We use Cartesian coordinates x_1,x_2,x_3 and consider our magnetofluid in the slab $0<x_1<1$ (that is, lengths are normalized to the thickness of the slab). In the x_2 and x_3 directions periodic boundary conditions, with spatial periods L_2 and L_3, are assumed.

In order to compensate for viscous and Ohmic losses and thus to admit nontrivial time-asymptotic states, there must be a net energy input through the boundary planes $x_1=0$, $x_1=1$. In the present paper we consider the case that only electromagnetic energy, in the form of a Poynting flux, can penetrate the boundary. In particular, we assume that there is no mass flow through the boundary, i.e.,

$$v_1=0 \quad \text{at } x_1=0,1. \quad (10)$$

With respect to the tangential velocity components, stress-free boundary conditions are used,

$$\frac{\partial v_2}{\partial x_1}=\frac{\partial v_3}{\partial x_1}=0 \quad \text{at } x_1=0,1. \quad (11)$$

The system is forced by applying an electric field of strength E^* in the x_3 direction. Of course, $\{\mathbf{E}\}$ can be prescribed only on the boundary, while in the interior of the volume considered it is determined by the governing equations. We further assume that there is no magnetic flux through the boundary,

$$B_1=0 \quad \text{at } x_1=0,1. \quad (12)$$

Conditions (10) and (12) imply that the tangential components of $\boldsymbol{v}\times\boldsymbol{B}$ on the boundary planes vanish, so that according to Eq. (9)

$$J_2=0, \quad J_3=E^* \quad \text{at } x_1=0,1. \quad (13)$$

The boundary conditions for the tangential components of $\boldsymbol{B}$ then become

$$\frac{\partial B_2}{\partial x_1}=E^*, \quad \frac{\partial B_3}{\partial x_1}=0 \quad \text{at } x_1=0,1. \quad (14)$$

A few remarks concerning the suitability and physical realizability of our boundary conditions seem to be in order (needless to say, we are considering a strongly idealized model). If there are rigid walls at $x_1=0$ and $x_1=1$, no-slip boundary conditions on the velocity ($\boldsymbol{v}=\boldsymbol{0}$) are of course more appropriate than stress-free ones. Stress-free boundaries are commonly assumed in order to circumvent the formation of viscous boundary layers (and thus to avoid the need to resolve small spatial scales). Now the main result of the present study will be the stability of a quiescent basic state. In this respect, stress-free boundaries are more general than rigid walls since the latter, by impeding fluid motions, are stabilizing. On the other hand, there are physical situations to which stress-free boundary conditions are actually well suited, notably in astrophysics. For instance, plasma loops and prominences in the solar corona are surrounded by a very tenuous plasma exerting practically no mechanical stresses on them (of course, in more realistic models also the deformation of free surfaces should be taken into account).

Still more delicate than the mechanical are the electromagnetic boundary conditions (cf. the discussion in Ref. [31]). The vanishing of the normal component of the magnetic field on the boundary planes [Eq. (12)] is most easily ensured by placing perfectly conducting rigid walls at $x_1=0$ and $x_1=1$. In this case, however, also the tangential component of the electric field has to vanish there (so that there is no Poynting flux through the boundary). In toroidal pinch devices in the laboratory, gaps in the (highly conducting) shell permit electric fields (as well as externally generated magnetic fields) to penetrate into the plasma, a situation that needs to be idealized to allow mathematical treatment. Shan, Montgomery, and Chen [25–29], who use boundary conditions slightly different from ours, namely, vanishing normal components of velocity, vorticity $(\boldsymbol{\nabla}\times\boldsymbol{v})$, magnetic field, and electric current density, idealize the boundary by a perfectly conducting wall coated inside with a thin layer of insulating dielectric. Our boundary conditions can be approximately realized if the wall is simply finitely conducting (and uncoated): Provided the homogeneous tangential electric field ($E_2=0$, $E_3=E^*$; E_1 is *not* presribed) can somehow be maintained in the wall, then the normal component of the magnetic field is independent of time [since $\partial B_1/\partial t=-(\partial E_3/\partial x_2-\partial E_2/\partial x_3)$], so that one merely has to ensure that it vanishes initially. The main difficulty, then, is to maintain the electric field at the boundary. In the laboratory the external electric field is usually provided inductively, which is possible only for a limited time. This time has to be long enough to allow the fluid or plasma to relax to its time-asymptotic state, in which we are primarily interested. Alternatively, since the two infinite plane walls have to be finite in reality, voltage drops could be directly applied between opposite edges; also the use of an array of thin electrodes held on potential values increasing linearly with x_3 is conceivable. The imposed tangential electric field leads to a tangential current in the wall, which in turn generates a magnetic field whose component normal to the wall vanishes.

Finally, the magnetic field in the fluid may contain a dc component, namely, a homogeneous field parallel to the boundaries. For our boundary conditions, this dc component is independent of time (cf. Sec. III and the Appendix) and thus a relic of the formation phase of the pinch. If a certain dc field is desired, the formation process has to be managed such as to generate it. For instance, applying first at one of the two boundaries a tangential electric field in the x_2 direction and then at both boundaries the permanent one in the x_3 direction will result in the presence of a dc magnetic field in the x_3 direction.

Our boundary conditions differ from those utilized by Shan, Montgomery, and Chen [25–29]. Their condition on the vorticity (the vanishing of its normal component) is implied by, but does not imply, no-slip boundary conditions; so this condition seems to be intermediate between stress-free and no-slip conditions. The vanishing of the normal components of magnetic field and electric current density required by Shan *et al.* still permits them to impose a (mean) tangential electric field at the boundary. So the driving mechanism is the same as in our case. We have preferred to assume conducting boundaries, which allow currents to flow normal to the boundary. But $|J_1|$ will be small if the conducting wall is thin and, say, separates the fluid from a vacuum or an insulating dielectric; then the two kinds of boundary conditions should be equally appropriate. It seems unlikely to us that the slight differences in the boundary conditions play a role for the observed fundamental difference between our results and those of Shan, Montgomery, and Chen [25–29].

There exists, up to a constant magnetic field, a unique stationary state with the fluid at rest: For this case Eqs. (5) and (6) simplify to the equations

$$-\boldsymbol{\nabla}p+(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}=\boldsymbol{0}, \tag{15}$$

$$\nabla^2\boldsymbol{B}=\boldsymbol{0}, \tag{16}$$

of which the last one, in connection with the boundary conditions given by Eqs. (12) and (14), implies

$$\boldsymbol{B}=(0,E^*x_1+C_2,C_3), \tag{17}$$

with constants C_2 and C_3. Thus the static equilibrium field $\boldsymbol{B}^e$, can be written as

$$\boldsymbol{B}^e=(0,E^*x_1-E^*/2+\overline{B_2},\overline{B_3}), \tag{18}$$

where overbars denote spatial averages over the periodicity volume, $0<x_1<1$, $0<x_2<L_2$, and $0<x_3<L_3$. The equilibrium current is uniform and in the x_3 direction,

$$\boldsymbol{J}^e=\boldsymbol{\nabla}\times\boldsymbol{B}^e=(0,0,E^*), \tag{19}$$

and there is a Lorentz force in the x_1 direction,

$$\boldsymbol{J}^e\times\boldsymbol{B}^e=(-B_2^eE^*,0,0). \tag{20}$$

Equation (15) is satisfied with

$$p=p^e=-\frac{\boldsymbol{B}^{e2}}{2}. \tag{21}$$

Obviously we could allow for a mean flow $\boldsymbol{v}^e=(0,\overline{v_2},\overline{v_3})$ in the equilibrium state ($\overline{v_1}$ has to vanish as a consequence of the boundary conditions in conjunction with the incompressibility): Eqs. (15)–(21) would remain valid; merely an electric field component $E_1=-(\boldsymbol{v}^e\times\boldsymbol{B}^e)_1$ would appear. But in a coordinate system comoving with the mean flow, we would again observe our static equilibrium.

We shall use the decomposition

$$(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}-\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{B}^2 \tag{22}$$

and write

$$P=p+\frac{1}{2}\boldsymbol{B}^2. \quad (23)$$

Furthermore, the notations

$$\boldsymbol{b}=\boldsymbol{B}-\boldsymbol{B}^e \quad (24)$$

and

$$\boldsymbol{j}=\boldsymbol{J}-\boldsymbol{E}^* \quad (25)$$

will be used. $\boldsymbol{v}$ and $\boldsymbol{b}$ will be our dynamical variables, for which the complete boundary conditions read

$$v_1=\frac{\partial v_2}{\partial x_1}=\frac{\partial v_3}{\partial x_1}=b_1=\frac{\partial b_2}{\partial x_1}=\frac{\partial b_3}{\partial x_1}=0 \quad \text{at } x_1=0,1. \quad (26)$$

The total energy flow S into the periodicity volume is given by

$$S=\int_0^{L_2}\int_0^{L_3}[(\boldsymbol{E}\times\boldsymbol{B})_1|_{x_1=0}-(\boldsymbol{E}\times\boldsymbol{B})_1|_{x_1=1}]dx_2dx_3$$
$$=E^*\int_0^{L_2}\int_0^{L_3}[B_2(x_1=1)-B_2(x_1=0)]dx_2dx_3$$
$$=E^*\int_0^{L_2}\int_0^{L_3}[b_2(x_1=1)-b_2(x_1=0)]dx_2dx_3+E^{*2}L_2L_3. \quad (27)$$

The term $E^{*2}L_2L_3$ just compensates for the Ohmic losses, given by $\int_0^1\int_0^{L_2}\int_0^{L_3}\boldsymbol{J}^2dx_1dx_2dx_3$, in the static equilibrium.

III. NUMERICAL METHOD, CALCULATIONS, AND RESULT

The boundary conditions given by Eq. (26) can be satisfied by expanding v_1 and b_1 in pure sine series and v_2, v_3, b_2, and b_3 in pure cosine series with respect to x_1; with respect to x_2 and x_3 expansions into exponential functions can be used:

$$v_1=\sum_{\boldsymbol{k}} v_{1\boldsymbol{k}}\sin(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$
$$v_2=\sum_{\boldsymbol{k}} v_{2\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$
$$v_3=\sum_{\boldsymbol{k}} v_{3\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$
$$b_1=\sum_{\boldsymbol{k}} b_{1\boldsymbol{k}}\sin(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$
$$b_2=\sum_{\boldsymbol{k}} b_{2\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$
$$b_3=\sum_{\boldsymbol{k}} b_{3\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\}. \quad (28)$$

Here $\boldsymbol{k}=(k_1,k_2,k_3)$ with

$$k_1=0,\pi,2\pi,3\pi,\ldots,$$
$$k_2=0,\mp\frac{2\pi}{L_2},\mp 2\frac{2\pi}{L_2},\mp 3\frac{2\pi}{L_2},\ldots$$
$$k_3=0,\mp\frac{2\pi}{L_3},\mp 2\frac{2\pi}{L_3},\mp 3\frac{2\pi}{L_3},\ldots \quad . \quad (29)$$

In Fourier space Eqs. (5) and (6) take the form

$$\dot v_{1\boldsymbol{k}}=-(\boldsymbol{v}\cdot\nabla v_1)_{\boldsymbol{k}}+k_1P_{\boldsymbol{k}}-P_m\boldsymbol{k}^2v_{1\boldsymbol{k}}+(\boldsymbol{B}\cdot\nabla B_1)_{\boldsymbol{k}},$$
$$\dot v_{2\boldsymbol{k}}=-(\boldsymbol{v}\cdot\nabla v_2)_{\boldsymbol{k}}-ik_2P_{\boldsymbol{k}}-P_m\boldsymbol{k}^2v_{2\boldsymbol{k}}+(\boldsymbol{B}\cdot\nabla B_2)_{\boldsymbol{k}},$$
$$\dot v_{3\boldsymbol{k}}=-(\boldsymbol{v}\cdot\nabla v_3)_{\boldsymbol{k}}-ik_3P_{\boldsymbol{k}}-P_m\boldsymbol{k}^2v_{3\boldsymbol{k}}+(\boldsymbol{B}\cdot\nabla B_3)_{\boldsymbol{k}},$$
$$\dot b_{1\boldsymbol{k}}=[\nabla\times(\boldsymbol{v}\times\boldsymbol{B})]_{1\boldsymbol{k}}-\boldsymbol{k}^2b_{1\boldsymbol{k}},$$
$$\dot b_{2\boldsymbol{k}}=[\nabla\times(\boldsymbol{v}\times\boldsymbol{B})]_{2\boldsymbol{k}}-\boldsymbol{k}^2b_{2\boldsymbol{k}},$$
$$\dot b_{3\boldsymbol{k}}=[\nabla\times(\boldsymbol{v}\times\boldsymbol{B})]_{3\boldsymbol{k}}-\boldsymbol{k}^2b_{3\boldsymbol{k}}. \quad (30)$$

In these equations the Fourier coefficients $P_{\boldsymbol{k}}$, $(\boldsymbol{v}\cdot\nabla v_i)_{\boldsymbol{k}}$, $(\boldsymbol{B}\cdot\nabla B_i)_{\boldsymbol{k}}$, and $[\nabla\times(\boldsymbol{v}\times\boldsymbol{B})]_{i\boldsymbol{k}}$ on the right-hand sides refer to expansions similar to those in Eq. (28), namely, of $\boldsymbol{v}\cdot\nabla v_1$, $\boldsymbol{B}\cdot\nabla B_1$, and $[\nabla\times(\boldsymbol{v}\times\boldsymbol{B})]_1$ in pure sine series and of P, $\boldsymbol{v}\cdot\nabla v_2$, $\boldsymbol{v}\cdot\nabla v_3$, $\boldsymbol{B}\cdot\nabla B_2$, $\boldsymbol{B}\cdot\nabla B_3$, $[\nabla\times(\boldsymbol{v}\times\boldsymbol{B})]_2$, and $[\nabla\times(\boldsymbol{v}\times\boldsymbol{B})]_3$ in pure cosine series with respect to x_1 (see the Appendix). Note that $\boldsymbol{B}$ (and not only $\boldsymbol{b}$) appears on the right-hand sides of the system (30), which reflects the forcing by the boundary electric field.

Equation (3) implies

$$k_1v_{1\boldsymbol{k}}+ik_2v_{2\boldsymbol{k}}+ik_3v_{3\boldsymbol{k}}=0, \quad (31)$$

$$k_1b_{1\boldsymbol{k}}+ik_2b_{2\boldsymbol{k}}+ik_3b_{3\boldsymbol{k}}=0, \quad (32)$$

so that $v_{1\boldsymbol{k}}$ can be expressed in terms of $v_{2\boldsymbol{k}}$ and $v_{3\boldsymbol{k}}$ and $b_{1\boldsymbol{k}}$ in terms of $b_{2\boldsymbol{k}}$ and $b_{3\boldsymbol{k}}$ (there are no complications in the case $k_1=0$ since $v_{1\boldsymbol{k}}$ and $b_{1\boldsymbol{k}}$ then vanish). Furthermore, by using the time derivative of Eq. (31),

$$k_1\dot{v}_{1\boldsymbol{k}}+ik_2\dot{v}_{2\boldsymbol{k}}+ik_3\dot{v}_{3\boldsymbol{k}}=0, \tag{33}$$

in conjunction with the first three equations of the system (30), the pressure can be eliminated: One obtains

$$\begin{aligned} P_{\boldsymbol{k}}=\frac{1}{\boldsymbol{k}^2}\{&k_1[(\boldsymbol{v}\cdot\boldsymbol{\nabla}v_1)_{\boldsymbol{k}}-(\boldsymbol{B}\cdot\boldsymbol{\nabla}B_1)_{\boldsymbol{k}}]\\ &+ik_2[(\boldsymbol{v}\cdot\boldsymbol{\nabla}v_2)_{\boldsymbol{k}}-(\boldsymbol{B}\cdot\boldsymbol{\nabla}B_2)_{\boldsymbol{k}}]\\ &+ik_3[(\boldsymbol{v}\cdot\boldsymbol{\nabla}v_3)_{\boldsymbol{k}}-(\boldsymbol{B}\cdot\boldsymbol{\nabla}B_3)_{\boldsymbol{k}}]\}. \end{aligned} \tag{34}$$

We have numerically studied the resulting system of ordinary differential equations (ODEs) for the unknown functions $v_{2\boldsymbol{k}}(t)$, $v_{3\boldsymbol{k}}(t)$, $b_{2\boldsymbol{k}}(t)$, and $b_{3\boldsymbol{k}}(t)$ by means of a pseudospectral method [32,33]. The nonlinear terms (products) on the right-hand sides were calculated in real space (instead of in Fourier space). This did not merely save computer time but really made feasible the calculations. The right-hand side of the first equation of the system (30), for instance, has to be expanded into a pure sine series with respect to x_1. However, if v_1 is given in the form of such a series and v_2 and v_3 are correspondingly given in the form of pure cosine series, then, by directly calculating in Fourier space, $\boldsymbol{v}\cdot\boldsymbol{\nabla}v_1$ becomes the sum of different products of sine and cosine functions (which have to be expanded into sine series). These difficulties are circumvented by Fourier transforming after having calculated the products in real space (for further details see the Appendix).

It can easily be shown that the spatial means of v_2, v_3, B_2, and B_3 are independent of time (cf. the Appendix). Without loss of generality we have restricted ourselves to the case of $\overline{v_2}=\overline{v_3}=0$ (that is, of vanishing Fourier coefficients $v_{2\boldsymbol{0}}$ and $v_{3\boldsymbol{0}}$) for, as noted in Sec. II, the mean flow can be removed by a Galilean transformation. The mean values $\overline{B_2}$ and $\overline{B_3}$, on the other hand, have been treated as parameters.

Keeping fixed P_m, L_2, L_3, $\overline{B_2}$, and $\overline{B_3}$ and increasing E^*, we have traced the static equilibrium solution. In each step of the tracing, in order to detect bifurcation points, the eigenvalues of the Jacobian matrix of our system of ODEs were calculated. The surprising result was that none of the real parts of the eigenvalues became positive (or at least zero), no matter how strong E^* and irrespective of the choice of the other (fixed) parameters. Of course, we could not systematically explore the space of these latter parameters. But they were varied in a broad range. P_m, in particular, was varied between 0 and 10^6. For L_2 and L_3 the values 1, 4, and 10 were selected, for $\overline{B_3}$ the values 0, 1, 10, and 1000, and for $\overline{B_2}$, finally, the values 0 and 10. In each tracing, E^* was increased up to a value of 10^6.

Because of the large amount of computer main storage needed for the Jacobian matrix, the eigenvalue calculations were restricted to a resolution of 16^3 grid points in real space. But we have also *simulated* the system for randomly chosen initial conditions. In the simulations, we could use a resolution of $64\times32\times32$ grid points, with the higher resolution in the direction of the equilibrium-field gradient (x_1). To test for aliasing errors [32,33], we also used a dealiased version of the subprogram calculating the right-hand sides of the system of ODEs. For the dealiasing the 2/3 rule was employed, namely, for each spatial direction one-third of the Fourier coefficients (those with the largest wave numbers) was set equal to zero before each transformation from Fourier to real space (where then the nonlinearities were calculated). In this way possible aliasing errors can be removed, but at the expense of a significantly harder truncation: in our case the number of active modes was reduced to about one-third (8/27). In the simulations, differences between the simple and dealiased calculations were observable for $E^*\gtrsim10^5$, but these did not affect the time-asymptotic state. For the numerical calculation of the elements of the Jacobian matrix, on the other hand, we could use a linearized version of the subprogram for the right-hand sides [in which, for example, the term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ was omitted]. Sources of aliasing errors are solely the product terms. In our case these were not completely removed by the linearization: the products of $\{\boldsymbol{v}\}$ and $\boldsymbol{b}$ with the equilibrium magnetic field $\boldsymbol{B}^e$ survived the linearization. However, the eigenvalues with the largest real parts, decisive for the stability, did not seem to be influenced by aliasing.

In all simulations, independent of the initial conditions and of the values of the parameters, asymptotically in time the static equilibrium $\boldsymbol{v}=\boldsymbol{b}=\boldsymbol{0}$ was approached. This confirms the stability of the basic state. Furthermore, it indicates that coexisting attractors not bifurcating from the basic state do not exist.

IV. CONCLUSION

In a voltage-driven incompressible sheet pinch with spatially and temporally uniform kinematic viscosity and magnetic diffusivity and with impenetrable stress-free boundaries, the quiescent basic state with uniform current density is absolutely stable. Furthermore, it seems to be the only attractor of the system, though this cannot be stated with the same confidence. We suppose the result obtained to be equally valid under rigid-wall boundary conditions for the velocity.

The complete absence of magnetohydrodynamic activity in the sheet pinch contrasts with the rich activity observed in corresponding numerical studies of the cylindrical pinch [25]. It seems unlikely to us that this results from the slight difference between the boundary conditions utilized in both studies. The situation is reminiscent of that for the hydrodynamic Couette flow [34–36]. Namely, for the plane Couette flow, the flow between infinite parallel planes with one moving boundary, the basic state with a linear velocity profile is stable, while in the rotating Couette flow, the flow between concentric cylinders with the inner cylinder rotating, various bifurcations are observed. As a next step, we plan to study the bifurcation properties of a sheet pinch with spatially nonuniform magnetic diffusivity.

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APPENDIX

The Fourier coefficients $P_{\boldsymbol{k}}$, $(\boldsymbol{v}\cdot\boldsymbol{\nabla}v_i)_{\boldsymbol{k}}$, $(\boldsymbol{B}\cdot\boldsymbol{\nabla}B_i)_{\boldsymbol{k}}$, $[\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})]_{i\boldsymbol{k}}$ on the right-hand sides of the system (30) are defined by the expansions

$$P=\sum_{\boldsymbol{k}} P_{\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_1=\sum_{\boldsymbol{k}} (\boldsymbol{v}\cdot\boldsymbol{\nabla}v_1)_{\boldsymbol{k}}\sin(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_2=\sum_{\boldsymbol{k}} (\boldsymbol{v}\cdot\boldsymbol{\nabla}v_2)_{\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_3=\sum_{\boldsymbol{k}} (\boldsymbol{v}\cdot\boldsymbol{\nabla}v_3)_{\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}B_1=\sum_{\boldsymbol{k}} (\boldsymbol{B}\cdot\boldsymbol{\nabla}B_1)_{\boldsymbol{k}}\sin(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}B_2=\sum_{\boldsymbol{k}} (\boldsymbol{B}\cdot\boldsymbol{\nabla}B_2)_{\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}B_3=\sum_{\boldsymbol{k}} (\boldsymbol{B}\cdot\boldsymbol{\nabla}B_3)_{\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$R_1=\sum_{\boldsymbol{k}} R_{1\boldsymbol{k}}\sin(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$R_2=\sum_{\boldsymbol{k}} R_{2\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\},$$

$$R_3=\sum_{\boldsymbol{k}} R_{3\boldsymbol{k}}\cos(k_1x_1)\exp\{i(k_2x_2+k_3x_3)\}, \qquad \text{(A1)}$$

where $\boldsymbol{R}$ is an abbreviation for $\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})$. For evaluating the terms $(\boldsymbol{v}\cdot\boldsymbol{\nabla}v_i)_{\boldsymbol{k}}$ and $(\boldsymbol{B}\cdot\boldsymbol{\nabla}B_i)_{\boldsymbol{k}}$, it has been advantageous to use the relations

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_1=\frac{\partial}{\partial x_1}v_1^2+\frac{\partial}{\partial x_2}(v_1v_2)+\frac{\partial}{\partial x_3}(v_1v_3),$$

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_2=\frac{\partial}{\partial x_2}v_2^2+\frac{\partial}{\partial x_1}(v_1v_2)+\frac{\partial}{\partial x_3}(v_2v_3),$$

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_3=\frac{\partial}{\partial x_3}v_3^2+\frac{\partial}{\partial x_1}(v_1v_3)+\frac{\partial}{\partial x_2}(v_2v_3) \qquad \text{(A2)}$$

and the analogous relations for $\boldsymbol{B}\cdot\boldsymbol{\nabla}B_i$. The products v_iv_j, and B_iB_j have been calculated in real space, Fourier transformed, and differentiated in Fourier space. Similarly, the terms $[\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})]_{i\boldsymbol{k}}$ were calculated.

Equation (A2) shows that the spatial means of the $\boldsymbol{v}\cdot\boldsymbol{\nabla}v_i$ vanish (by virtue of the boundary conditions). The same applies to the $\boldsymbol{B}\cdot\boldsymbol{\nabla}B_i$. Thus, according to Eq. (30), the spatial means of v_2 and v_3 (namely, the Fourier coefficients $v_{2\boldsymbol{0}}$ and $v_{3\boldsymbol{0}}$) are independent of time.

Correspondingly, the relations

$$[\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})]_2=\frac{\partial}{\partial x_3}(v_2B_3-v_3B_2)-\frac{\partial}{\partial x_1}(v_1B_2-v_2B_1),$$

$$[\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})]_3=\frac{\partial}{\partial x_1}(v_3B_1-v_1B_3)-\frac{\partial}{\partial x_2}(v_2B_3-v_3B_2) \qquad \text{(A3)}$$

show that the mean values of $\overline{[\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})]_2}$ and $\overline{[\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})]_3}$ vanish, so that according to Eq. (30), $\overline{b_2}$ and $\overline{b_3}$ are independent of time.

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